Stochastic Submodular Maximization via Polynomial Estimators

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Abstract. In this paper, we study stochastic submodular maximization problems with general matroid constraints, which naturally arise in online learning, team formation, facility location, influence maximization, active learning and sensing objective functions. In other words, we focus on maximizing submodular functions that are defined as expectations over a class of submodular functions with an unknown distribution. We show that for monotone functions of this form, the stochastic continuous greedy algorithm [20] attains an approximation ratio (in expectation) arbitrarily close to $(1 - 1/e) \approx 63\%$ using a polynomial estimation of the gradient. We argue that using this polynomial estimator instead of the prior art that uses sampling eliminates a source of randomness and experimentally reduces execution time.

Keywords: submodular maximization · stochastic optimization · greedy algorithm.

1 Introduction

Submodular maximization is a true workhorse of data mining, arising in settings as diverse as hyper-parameter optimization [25], feature compression [3], text classification [15], and influence maximization [9, 14]. Many of these interesting problems as well as variants can be cast as maximizing a submodular set function $f(S)$, defined over sets $S \subseteq V$ for some ground set V, subject to a matroid constraint. Despite the NP-hardness of these problems, the so-called *continuous-greedy* (CG) algorithm [5], can be used to construct a $1 - 1/e$-approximate solution in polynomial time. Interestingly, the solution is generated by first transferring the problem to the continuous domain, and solving a continuous optimization problem via gradient techniques. The solution to this continuous optimization problem is subsequently rounded (via techniques such as pipage rounding [1] and swap rounding [5]), to produce an integral solution within a $1 - 1/e$ factor from the optimal. The continuous optimization problem solved by the CG algorithm amounts to maximizing so-called *multilinear relaxation* of the original, combinatorial submodular objective. In short, the multilinear relaxation of a submodular function $f(S)$ is its expectation assuming its input S is generated via independent Bernoulli trials, and is typically computed via sampling [5, 26].

Recently, a series of papers have studied an interesting variant called the *stochastic submodular optimization* setting [2, 7, 11, 12, 20, 28]. In this setting, the submodular objective function to be optimized is assumed to be of the form of an expectation, i.e., $f(S) = \mathbb{E}_{z\sim P}[f_z(S)]$, where z is a random variable. Moreover, the optimization algorithm does not have access to the a function oracle (i.e., cannot compute the function itself). Instead it can only sample a random instantiation of $f_z(\cdot)$, different each time. This setting is of course of interest when the system or process that f models is inherently stochastic (e.g., involves a system dependent on, e.g., user behavior or random arrivals) and the distribution governing this distribution is not a priori known. It is also of interest when the support of distribution P is very large, so that the expectation cannot be computed efficiently. A classic example of the latter case is influence maximization (c.f. Sec. 3.1), where the expectation $f(S)$ cannot be computed efficiently or even in a closed form, even though samples $z \sim P$ can be drawn.

Interestingly, the fact that the classic continuous greedy algorithm operates in the continuous domain gives rise to a *stochastic continuous greedy* (SCG) method for tackling the stochastic optimization problem [20]. In a manner very similar to stochastic gradient descent, the continuous greedy algorithm can be modified to use *stochastic gradients*, i.e., random variables whose expectations equal the gradient of the multilinear relaxation. In practice, these are computed by sampling *two random variables in tandem*: $z \sim P$, which is needed to generate a random instance f_z, and S, the random input needed to compute the multilinear relaxation. As a result, the complexity of the SCG algorithm depends on the variance due to *both* of these two variables.

We make the following contributions:

- We use polynomial approximators, originally proposed by Özcan et al. [23], to reduce the variance of the stochastic continuous greedy algorithm. In particular, we eliminate one of the two sources of randomness of SCG, namely, sampling S. We do this by replacing the sampling estimator by a deterministic estimator constructed by approximating each $f_z(\cdot)$ with a polynomial function.
- We show that doing so *reduces the variance* of the gradient estimation procedure used by SCG, but introduces a *bias*. We then characterize the performance of SCG in terms of both the (reduced) variance and new bias term.
- We show that for several interesting stochastic submodular maximization problems, including influence maximization, the bias can be well-controlled, decaying exponentially with the degree of our polynomial approximators.
- Finally, we illustrate the advantage of our approach experimentally, over both synthetic and real-life datasets.

2 Related Work

While submodular optimization problems are generally NP-hard, the celebrated greedy algorithm [21] attains a $(1 - 1/e)$ approximation ratio for submodular maximization subject to uniform matroids and a $1/2$ approximation ratio for general matroid constraints. As discussed in the introduction, the continuous greedy algorithm [5] restores the $(1 - 1/e)$ approximation ratio by lifting the discrete problem to the continuous domain via the multilinear relaxation.

Stochastic submodular maximization, in which the objective is expressed as an expectation, has gained a lot of interest in the recent years [2,7,28]. Karimi et al. [12] use a concave relaxation method that achieves the $(1-1/e)$ approximation guarantee, but only for the class of submodular coverage functions. Hassani et al. [11] provide projected gradients methods for the general case of stochastic submodular problems that achieve $1/2$ approximation guarantee. Mokhtari et al. [20] propose stochastic conditional gradient methods for solving both minimization and maximization stochastic submodular optimization problems. Their method for maximization, Stochastic Continous Greedy (SCG) can be interpreted as a stochastic variant of the continuous greedy algorithm [5, 26] and achieves a tight $(1 - 1/e)$ approximation guarantee for monotone and submodular functions.

Our work builds upon and relies on the approach by Özcan et al. [23], who studied ways of accelerating the computation of gradients via a polynomial estimator. Extending on the work of

Mahdian et al. [17], Özcan et al. show that submodular functions that can be written as compositions of (a) an analytic function and (b) a multilinear function can be arbitrarily well approximated via Taylor polynomials; in turn, this gives rise to a method for approximating their multilinear relaxation in a closed form, without sampling. We leverage this method in the context of stochastic submodular optimization, showing that it can also be applied in combination with SCG of Mokhtari et al. [20]: this eliminates one of the two sources of randomness, thereby reducing variance at the expense of added bias. From a technical standpoint, this requires controlling the error introduced by the bias of the polynomial estimator, while simultaneously accounting for the variance inherent in SCG, due to sampling instances.

3 Technical Preliminary

Submodularity and Matroids. Given a ground set $V = \{1, \dots, n\}$ of n elements, a set function $f : 2^V \to \mathbb{R}_+$ is submodular if and only if $f(B \cup \{e\}) - f(B) \leq f(A \cup \{e\}) - f(A)$, for all $A \subseteq B \subseteq V$ and $e \in V$. Function f is *monotone* if $f(A) \leq f(B)$, for every $A \subseteq B$.
Matroids. Given a ground set V, a matroid is a pair $\mathcal{M} = (V, \mathcal{I})$, where $\mathcal{I} \subseteq 2^V$ is a collection of *independent sets*, for which the following hold: (a) if $B \in \mathcal{I}$ and $A \subset B$, then $A \in \mathcal{I}$, and (b) if $A, B \in \mathcal{I}$ and $|A| < |B|$, there exists $x \in B \setminus A$ s.t. $A \cup \{x\} \in \mathcal{I}$. The *rank* of a matroid $r_{\mathcal{M}}(V)$ is the largest cardinality of its elements, i.e.: $r_{\mathcal{M}}(V) = \max\{|A| : A \in \mathcal{I}\}$. We introduce two examples of matroids:

1. **Uniform Matroids.** The uniform matroid with cardinality k is $\mathcal{I} = \{S \subseteq V, |S| \leq k\}$.
2. **Partition Matroids.** Let $\mathcal{B}_1, \dots, \mathcal{B}_m \subseteq V$ be a partitioning of V, i.e., $\bigwedge_{\ell=1}^m \mathcal{B}_\ell = \emptyset$ and $\bigcup_{\ell=1}^m \mathcal{B}_\ell = V$. Let also $k_\ell \in \mathbb{N}, \ell = 1, \dots, m$, be a set of cardinalities. A partition matroid is defined as $\mathcal{I} = \{S \subseteq 2^V \mid |S \cap \mathcal{B}_\ell| \leq k_\ell, \text{ for all } \ell = 1, \dots, m\}$.

3.1 Problem Definition

In this work, we focus on *discrete stochastic submodular maximization* problems. More specifically, we consider set function $f : 2^V \to \mathbb{R}_+$ of the form: $f(S) = \mathbb{E}_{z \sim P}[f_z(S)]$, $S \subseteq V$, where z is the realization of the random variable Z drawn from a distribution P over a probability space (V_z, P). For each realization of $z \sim P$, the set function $f_z : 2^V \to \mathbb{R}_+$ is monotone and submodular. Hence, f itself is monotone and submodular. The objective is to maximize f subject to some constraints (e.g., cardinality or matroid constraints) by only accessing to i.i.d. samples of $f_{z \sim P}$. In other words, we wish to solve:

$$\max_{S \in \mathcal{I}} f(S) = \max_{S \in \mathcal{I}} \mathbb{E}_{z \sim P}[f_z(S)], \tag{1}$$

where $\mathcal{I}$ is a general matroid constraint.

Stochastic submodular maximization problems are of interest in the absence of the oracle that provides the exact value of $f(S)$: one can only access $f_z(S)$, for random instantiations $z \sim P$. A well-known motivational example is contagion propagation in a network (a.k.a., the influence maximization problem [14]). Given a graph with node set V, the reachability of nodes from seeds is determined by sampling sub-graph $G = (V, E)$, via, e.g., the Independent Cascade or the Linear Threshold model [14]. The random edge set, in this case, plays the role of z, and the distribution over graphs the role of P. The function $f_z(S)$ represents the ratio of nodes reachable from the seeds S under the connectivity induced by edges E in this particular realization of z. The goal is to select seeds S that maximize $f(S) = \mathbb{E}_{z\sim P}[f_z(S)]$; both f and f_z are monotone submodular functions; however computing f in a closed form is hard, and $f(\cdot)$ can only be accessed through random instantiations of $f_z(\cdot)$.

3.2 Change of Variables and Multiliear Relaxation

There is a 1-to-1 correspondence between a binary vector $\mathbf{x} \in \{0,1\}^n$ and its support $S = \texttt{supp}(\mathbf{x})$. Hence, a set function $f : 2^V \to \mathbb{R}_+$ can be interpreted as $f : \{0,1\}^n \to \mathbb{R}_+$ via: $f(\mathbf{x}) \triangleq f(\texttt{supp}(\mathbf{x}))$ for $\mathbf{x} \in \{0,1\}^n$. We adopt this convention for the remainder of the paper. We also treat matroids as subsets of $\{0,1\}^n$, defined consistently with this change of variables via $\mathcal{M} = \{\mathbf{x} \in \{0,1\}^n : \text{supp}(\mathbf{x}) \in \mathcal{I}\}$. For example, a partition matroid is: $\mathcal{M} = \left\{\mathbf{x} \in \{0,1\}^n \mid \bigcap_{\ell=1}^m \left(\sum_{i\in B_\ell} x_i \leq k_\ell\right)\right\}$. The *matroid polytope* $\mathcal{C} \subseteq [0,1]^n$ is the convex hull of matroid $\mathcal{M}$, i.e., $\mathcal{C} = \texttt{conv}(\mathcal{M})$.

We define the *multilinear relaxation* of f as:

$$\begin{aligned} G(\mathbf{y}) &= \mathbb{E}_{S\sim\mathbf{y}}[f(S)] = \sum_{S\subseteq V} f(S) \prod_{i\in S} y_i \prod_{j\notin S}(1-y_j) \\ &= \mathbb{E}_{\mathbf{x}\sim\mathbf{y}}[f(\mathbf{x})] = \sum_{\mathbf{x}\in\{0,1\}^n} f(\mathbf{x}) \prod_{i\in V} y_i^{x_i}(1-y_i)^{(1-x_i)}, \quad \text{for } \mathbf{y} \in [0,1]^n. \end{aligned} \tag{2}$$

In other words, $G : [0,1]^n \to \mathbb{R}_+$ is the expectation of f, assuming that S is random and generated from independent Bernoulli trials: for every $i \in V$, $P(i \in S) = y_i$. The multilinear relaxation of f satisfies several properties. First, it is indeed a relaxation/extension of f over the (larger) domain $[0,1]^n$: for $\mathbf{x} \in \{0,1\}^n$, $G(\mathbf{x}) = f(\mathbf{x})$, i.e., G agrees with f on integral inputs. Second, it is *multilinear* (c.f. Sec. 3.4), i.e., affine w.r.t. any single coordinate y_i, $i \in V$, when keeping all other coordinates $\mathbf{y}_{-i} = [y_j]_{j\neq i}$ fixed. Finally, in the context of stochastic submodular optimization, it is an expectation that involves *two sources of randomness*: (a) $z \sim P$, i.e., the random instantiation of the objective, *as well as* (b) $\mathbf{x} \sim \mathbf{y}$, i.e., the independent sampling of the Bernoulli variables (i.e., the set S). In particular, we can write:

$$G(\mathbf{y}) = \mathbb{E}_{z\sim P}[G_z(\mathbf{y})], \text{ where } G_z(\mathbf{y}) = \mathbb{E}_{\mathbf{x}\sim\mathbf{y}}[f_z(x)] \text{ is the multilinear relaxation of } f_z(\cdot). \tag{3}$$

3.3 Stochastic Continuous Greedy Algorithm

The stochastic nature of the set function $f(S)$ requires the use the *Stochastic Continuous Greedy (SCG)* algorithm [20]. This is a stochastic variant of the continuous greedy algorithm (method) [26], to solve (1). The SCG algorithm uses a common averaging technique in stochastic optimization and computes the estimated gradient $\mathbf{d}_t$ by the recursion

$$\mathbf{d}_t = (1-\rho_t)\mathbf{d}_{t-1} + \rho_t \nabla G_{z_t}(\mathbf{y}_t), \tag{4}$$

where ρ_t is a positive step size and the algorithm initially starts with $\mathbf{d}_0 = \mathbf{y}_0 = \mathbf{0}$. Then, it proceeds in iterations, where in the t-th iteration it finds a feasible solution as follows

$$\mathbf{v}_t \in \arg\max_{\mathbf{v}\in\mathcal{C}}\{\mathbf{d}_t^T \mathbf{v}\}, \tag{5}$$

where $\mathcal{C}$ is the matroid polytope (i.e., convex hull) of matroid $\mathcal{M}$. After finding the ascent direction $\mathbf{v}_t$, the current solution $\mathbf{y}_t$ is updated as

$$\mathbf{y}_{t+1} = \mathbf{y}_t + \frac{1}{T}\mathbf{v}_t, \tag{6}$$

Algorithm 1 Stochastic Continuous Greedy (SCG)

Require: Step sizes $\rho_t > 0$. Initialize $\mathbf{d}_0 = \mathbf{y}_0 = \mathbf{0}$.
1: **for** $t = 1, 2, \ldots, T$ **do**
2: Compute $\mathbf{d}_t = (1-\rho_t)\mathbf{d}_{t-1} + \rho_t \nabla G_{z_t}(\mathbf{y}_t)$;
3: Compute $\mathbf{v}_t \in \arg\max_{\mathbf{v}\in\mathcal{C}}\{\mathbf{d}_t^T \mathbf{v}\}$;
4: Update the variable $\mathbf{y}_{t+1} = \mathbf{y}_t + \frac{1}{T}\mathbf{v}_t$;
5: **end for**

where $1/T$ is the step size. The steps of the stochastic continuous greedy algorithm are outlined in Algorithm 1. The (fractional) output of Algorithm 1 is within a $1-1/e$ factor from the optimal solution to Problem (1) (see Theorem 2 below). This fractional solution can subsequently be rounded in polynomial time to produce a solution with the same approximation guarantee w.r.t. to Problem (1) using, e.g., either the pipage rounding [1] or the swap rounding [6] methods.

Sample Estimator. The gradient ∇G_{z_t} is needed to perform step (4); computing it directly via Eq. (2). requires exponentially many calculations. Instead, both Calinescu et al. [5] and Mokhtari et al. [20] estimate it via *sampling*. In particular, due to multilinearity (i.e., the fact that G_z is affine w.r.t. a coordinate x_i, we have:

$$\frac{\partial G_z(\mathbf{y})}{\partial x_i} = G_z([\mathbf{y}]_{+i}) - G_z([\mathbf{y}]_{-i}), \quad \text{for all } i \in V, \tag{7}$$

where $[\mathbf{y}]_{+i}$ and $[\mathbf{y}]_{-i}$ are equal to the vector $\mathbf{y}$ with the i-th coordinate set to 1 and 0, respectively. The gradient of G can thus be estimated by (a) producing N random samples $\mathbf{x}^{(l)}$, for $l \in \{1,\ldots,N\}$ of the random vector $\mathbf{x}$, and (b) computing the empirical mean of the r.h.s. of (7), yielding

$$\frac{\partial \widehat{G_z(\mathbf{y})}}{\partial x_i} = \frac{1}{N}\sum_{l=1}^{N}\left(f_z([\mathbf{x}^{(l)}]_{+i}) - f_z([\mathbf{x}^{(l)}]_{-i})\right), \quad \text{for all } i \in V. \tag{8}$$

Mokhtari et al. [20] make the following assumptions:

Assumption 1. *Function $f : \{0,1\}^n \to \mathbb{R}_+$ is monotone and submodular.*

Assumption 2. *The Euclidean norm of the elements in the constraint set $\mathcal{C}$ are uniformly bounded, i.e., for all $\mathbf{y} \in \mathcal{C}$, there exists a D s.t. $\|\mathbf{y}\| \le D$.*

Under these assumptions, SCG combined with the sampling estimator in Eq. (7), yields the following guarantee:

Theorem 1. *[Mokhtari et al. [20]] Consider Stochastic Continuous Greedy (SCG) outlined in Algorithm 1, with $\nabla G_{z_t}(\mathbf{y}_t)$ replaced by $\nabla \widehat{G_{z_t}(\mathbf{y}_t)}$ given by (8). Recall the definition of the multilinear extension function G in (2) and set the averaging parameter as $\rho_t = 4/(t+8)^{2/3}$. If Assumptions 1 & 2 are satisfied, then the iterate $\mathbf{y}_T$ generated by SCG satisfies the inequality*

$$\mathbb{E}\left[G(\mathbf{y}_T)\right] \ge (1-1/e)OPT - \frac{15DK}{T^{1/3}} - \frac{f_{\max} r D^2}{2T}, \tag{9}$$

where $OPT = \max_{\mathbf{y}\in\mathcal{C}} G(\mathbf{y})$ *and* $K = \max\{3\|\nabla G(\mathbf{y}_0) - \mathbf{d}_0\|, 4\sigma + \sqrt{3r} f_{\max} D\}$, *where* D *is the diameter of the convex hull* $\mathcal{C}$, $f_{\max}$ *is the maximum marginal value of the function* f, *i.e.,* $f_{\max} = \max_{i\in\{1,\ldots,n\}} f(\{i\})$, r *is the rank of the matroid* $\mathcal{I}$, *and* $\sigma^2 = \sup_{\mathbf{y}\in\mathcal{C}} \mathbb{E}\left[\|\widehat{\nabla G_z(\mathbf{y})} - G(\mathbf{y})\|\right]$, *where* $\widehat{\nabla G_z}$ *is the sample estimator given by Eq.* (8).

Thus, by appropriately setting the number of iterations T, we can produce a solution that is arbitrarily close to $1 - 1/e$ from the optimal (fractional) solution. Again, this can be subsequently rounded (see, e.g., [1, 5]) to produce an integer solution with the same approximation guarantee. It is important to note that the number of steps required depends on σ^2, which is a (uniform over $\mathcal{C}$) bound on the variance of the estimator given by Eq. (8). This variance contains *two sources of randomness*, namely $z \sim P$, the random instantiation, and $\mathbf{x} \sim \mathbf{y}$, as multiple such integer vectors/sets are sampled in Eq (8). In general, the variance will depend on the number of samples N in the estimator, and will be bounded (as G is bounded).[1]

3.4 Multilinear Functions and the Multilinear Relaxation of a Polynomial

Recall that a *polynomial* function $p : \mathbb{R}^n \to \mathbb{R}$ can be written as a linear combination of several monomials, i.e.,

$$p(\mathbf{y}) = c_0 + \sum_{\ell\in\mathcal{I}} c_\ell \prod_{i\in\mathcal{J}_\ell} y_i^{k_i^\ell}, \tag{10}$$

where $c_\ell \in \mathbb{R}$ for ℓ in some index set $\mathcal{I}$, subsets $\mathcal{J}_\ell \subseteq V$ determine the terms of each monomial, and , and $\{k_i^\ell\}_{i\in\mathcal{J}_\ell} \subset \mathbb{N}$ are natural exponents. W.l.o.g. we assume that $k_i^\ell \geq 1$ (as variables with zero exponents can be ommited). The degree of the monomial indexed by $\ell \in \mathcal{I}$ is $k^\ell = \sum_{i\in\mathcal{J}_\ell} k_i^\ell$, and the degree of polynomial p is $\max_{\ell\in\mathcal{I}} k^\ell$, i.e., the largest degree across monomials.

A function $f : \mathbb{R}^N \to \mathbb{R}$ is *multilinear* if it is affine w.r.t. each of its coordinates [4]. Alternatively, multilinear functions are polynomial functions in which the degree of each variable in a monomial is at most 1; that is, multilinear functions can be written as:

$$f(\mathbf{y}) = c'_0 + \sum_{\ell\in\mathcal{I}} c'_\ell \prod_{i\in\mathcal{J}_\ell} y_i, \tag{11}$$

where $c_\ell \in \mathbb{R}$ for ℓ in some index set $\mathcal{I}$, and subsets $\mathcal{J}_\ell \subseteq V$, again determining monomials of degree *exactly equal to* $|\mathcal{J}_\ell|$. Given a polynomial p defined by the parameters in Eq. (10), let

$$\dot{p}(\mathbf{y}) = c_0 + \sum_{\ell\in\mathcal{I}} c_\ell \prod_{i\in\mathcal{J}_\ell} y_i, \tag{12}$$

be the multilinear function resulting from p, *by replacing all its exponents* $k_i^\ell \geq 1$ *with* 1. We call this function the *multilinearization* of p. The multilinearization of p is inherently linked to its multilinear relaxation:

Lemma 1 (Özcan et al. [23]). *Let* $p : [0,1]^n \to \mathbb{R}$ *be an arbitrary polynomial and let* $\dot{p} : \mathbb{R}^n \to \mathbb{R}_+$ *be its multilinearization, given by Eq.* (12). *Let* $\mathbf{x} \in \{0,1\}^n$ *be a random vector of independent Bernoulli coordinates parameterized by* $\mathbf{y} \in [0,1]^n$. *Then,* $\mathbb{E}_{\mathbf{x}\sim\mathbf{y}}[p(\mathbf{x})] = \mathbb{E}_{\mathbf{x}\sim\mathbf{y}}[\dot{p}(\mathbf{x})] = \dot{p}(\mathbf{y})$.

[1] For example, even for $N = 1$, the submodularity of f_z and Eq. (7) imply that $\sigma^2 \leq 2n\max_{j\in[n]} \mathbb{E}[f_z(\{j\})^2]$ [20], though this bound is loose/a worst-case bound.

Proof. Observe that $\dot{p}(\mathbf{x}) = p(\mathbf{x})$, for all $\mathbf{x} \in \{0,1\}^n$. This is precisely because $x^k = x$ for $x \in \{0,1\}$ and all $k \geq 1$. The first equality therefore follows. On the other hand, $\dot{p}(\mathbf{x})$ is the multilinear function given by Eq. (12). Hence $\mathbb{E}_{\mathbf{x}\sim\mathbf{y}}[\dot{p}(\mathbf{x})] = \mathbb{E}_{\mathbf{x}\sim\mathbf{y}}\left[c_0 + \sum_{\ell\in\mathcal{I}} c_\ell \prod_{i\in\mathcal{J}_\ell} x_i\right] = c_0 + \sum_{\ell\in\mathcal{I}} \mathbb{E}_{\mathbf{x}\sim\mathbf{y}}\left[\prod_{i\in\mathcal{J}_\ell} x_i\right] = c_0 + \sum_{\ell\in\mathcal{I}}\prod_{i\in\mathcal{J}_\ell} \mathbb{E}_{\mathbf{x}\sim\mathbf{y}}\left[x_i\right] = \dot{p}(\mathbf{y})$, where the second to last equality holds by the independence across x_i, $i \in V$. □

An immediate consequence of this lemma is that the multilinear relaxation of any polynomial function can be computed *without sampling*, by simply computing its multilinearization. This is of particular interest of course for submodular functions that are themselves polynomials (e.g., coverage functions [12]). Özcan et al. extend this to submodular functions that can be written as compositions of a scalar and a polynomial function, by approximating the former via its Taylor expansion. We extend and generalize this to the case of stochastic submodular functions, so long as the latter can be approximated arbitrarily well by polynomials.

4 Main Results

4.1 Polynomial Estimator

To leverage Lem. 1 to the case of stochastic submodular functions, we make the following assumption:

Assumption 3. *For all $z \in V_z$, there exists a sequence of polynomials $\{\hat{f}_z^L\}_{L=1}^{\infty}$, $\hat{f}_z^L : \mathbb{R}^n \to \mathbb{R}$ such that $\lim_{L\to\infty} |f_z(\mathbf{x}) - \hat{f}_z^L(\mathbf{x})| = 0$, uniformly over $\mathbf{x} \in \{0,1\}^n$, i.e. there exists $\varepsilon_z(L) \geq 0$ such that $\lim_{L\to\infty} \varepsilon_z(L) = 0$ and $|f_z(\mathbf{x}) - \hat{f}_z^L(\mathbf{x})| \leq \varepsilon_z(L)$, for all $\mathbf{x} \in \{0,1\}^n$.*

In other words, we assume that we can asymptotically approximate every function f_z with a polynomial arbitrarily well. Note that there already exists a polynomial function that approximates each f_z *perfectly* (i.e., $\epsilon_z = 0$), namely, its multilinear relaxation G_z. However, the number of terms in this polynomial is exponential in n. In contrast, Asm. 3 requires exact recovery only asymptotically. In many cases, this allows us to construct polynomials with only a handful (i.e., polynomial in n) terms, that can approximate f_z. We will indeed present such polynomials for several applications of interest in Section 5. Armed with this assumption, we define an estimator $\widehat{\nabla G_z^L}$ of the gradient of the multilinear relaxation G as follows:

$$\frac{\widehat{\partial G_z^L}}{\partial y_i}\Big|_{\mathbf{y}} \equiv \mathbb{E}_{\mathbf{x}\sim\mathbf{y}}[\hat{f}_z^L([\mathbf{x}]_{+i})] - \mathbb{E}_{\mathbf{x}\sim\mathbf{y}}[\hat{f}_z^L([\mathbf{x}]_{-i})] \overset{\text{Lem. 1}}{=} \dot{\hat{f}}_z^L([\mathbf{y}]_{+i}) - \dot{\hat{f}}_z^L([\mathbf{y}]_{-i}), \text{ for all } i \in V. \quad (13)$$

In other words, our estimator is constructed by replacing the multilinear relaxation G_z in Eq. (7) with the multilinear relaxation of the approximating polynomial $\hat{f}_z$. In turn, by Lem. 1, *the latter can be computed deterministically (without any sampling of the Bernoulli variables $\mathbf{x} \sim \mathbf{y}$)*, in closed form: the latter is given by the multilinearization $\dot{\hat{f}}_z^L$ of polynomial $\hat{f}_z^L$.

Nevertheless, our deterministic estimator given by Eq. (13) has a *bias*, precisely because of our approximation of f_z via the polynomial $\hat{f}_z^L$. We characterize this bias via the following lemma:

Lemma 2. *Assume that function f_z satisfies Asm. 3. Let ∇G_z be the unbiased stochastic gradient for a given f_z and let $\widehat{\nabla G_z^L}$ be the estimator of the multilinear relaxation given by* (13)*. Then, $\left\|\nabla G_z(\mathbf{y}) - \widehat{\nabla G_z^L}(\mathbf{y})\right\|_2 \leq 2\sqrt{n}\varepsilon_z(L)$, for all $\mathbf{y} \in \mathcal{C}$.*

The proof can be found in App. A. Hence, we can approximate ∇G arbitrarily well, uniformly over all $\mathbf{x} \in [0,1]^n$. We can thus use our estimator in the SCG algorithm instead of of the sample estimator of the gradient (Eq. (8)). We prove that this yields the following guarantee:

Theorem 2. *Consider Stochastic Continuous Greedy (SCG) outlined in Algorithm 1. Recall the definition of the multilinear extension function G in* (2)*. If Asm. 1 is satisfied and $\rho_t = 4/(t+8)^{2/3}$, then the objective function value for the iterates generated by SCG satisfies the inequality*

$$\mathbb{E}[G(\mathbf{y}_T)] \geq (1-1/e)OPT - \frac{15DK}{T^{1/3}} - \frac{f_{\max} r D^2}{2T},$$

where $K = \max\{3\|\nabla G(\mathbf{y}_0 - \mathbf{d}_0)\|^2, \sqrt{16\sigma_0^2 + 224\sqrt{n}\varepsilon(L)} + 2\sqrt{r} f_{\max} D\}$, $OPT = \max_{\mathbf{y}\in\mathcal{C}} G(\mathbf{y})$, *$r$ is the rank of the matroid $\mathcal{I}$,* $\varepsilon(L) = \mathbb{E}_{z\sim P}[\varepsilon_z(L)]$, *$f_{\max}$ is the maximum marginal value of the function f, i.e.,* $f_{\max} = \max_{i\in\{1,\dots,n\}} f(\{i\})$, *and* $\sigma_0^2 = \sup_{\mathbf{y}\in\mathcal{C}} \mathbb{E}_{z\sim P}\left[\|\nabla G(\mathbf{y}) - \nabla G_z(\mathbf{y})\|^2\right]$.

The proof can be found in App. B of [22]. Our proof follows the main steps of [20], using however the bias guarantee from Lem. 2; to do so, we need to deal with the fact that our estimator is not unbiased, but also that stochasticity is still present (as variables z are still sampled randomly). This is also reflected in our bound, that contains both a bias term (via $\varepsilon(L)$) and a variance term (via σ_0).

Comparing our guarantee to Thm. 1, we observe two main differences. On one hand, we have replaced the uniform bound of the variance σ^2 with the smaller quantity σ_0^2: the latter is quantifying the gradient variance w.r.t. z, and is thus smaller than σ, that depends on the variance of *both* z *and* $\mathbf{x} \sim \mathbf{y}$. Crucially, σ_0^2 is an "inherent" variance, *independent of the gradient estimation process*: it is the variance due to the randomness z, which is inherent in how we access our stochastic submodular objective and thus cannot be avoided. On the other hand, this variance reduction comes at the expense of introducing a bias term. This, however, can be suppressed via Asm. 3; as we discuss in the next section, for several problems of interest, this can be made arbitrarily small using only a polynomial number of terms in $\hat{f}_z^L$.

5 Problem Examples

In this section, we list several problems that can be tackled through our approach, also summarized in Tab. 1; these are similar to the problems considered by Özcan et al. [23], but cast into the stochastic submodular optimization setting. All problems correspond to trivially bounded variances σ_0^2 (again, because functions f_z are bounded); we thus focus on determining their bias $\epsilon(L)$. For space reasons, we report Cache Networks (CN) in Table 1, but provide details for it in the App. C.

5.1 Data Summarization (SM) [13, 16, 18]

In data summarization, ground set V is a set of tokens, representing, e.g., words or sentences in a document. A corpus of documents V_z is presented to us sequentially, and the goal is to select a "summary" $S \subseteq V$ that is representative of V_z. The summary should be simultaneously (a) representative of the corpus, and (b) diverse.

To be representative, the summary $S \subset V$ should contain tokens of high value, where the value of a token is document-dependent: for document $z \in V_z$, token $i \in V$ has a value $r_{i,z} \in [0,1]$,

Table 1. Summary of problems satisfying Asm. 1& 3.

	Input	$g_z : \{0,1\}^{\|V\|} \to [0,1]$ $\mathbf{x} \to g_z(\mathbf{x})$	$f_z : \{0,1\}^{\|V\|} \to \mathbb{R}_+$ $\mathbf{x} \to f_z(\mathbf{x})$	$\hat{f}_z^L : \{0,1\}^{\|V\|} \to \mathbb{R}_+$ $\mathbf{x} \to \hat{f}_z^L(\mathbf{x})$	Bias $\varepsilon(L)$
SM	Weighted bipartite graph $G = (V \cup P)$ weights $\mathbf{r}_z \in \mathbb{R}_+^n$, and $\sum_{i=1}^n r_{i,z} = 1$	$\sum_{i \in V \cap P_j} r_{i,z} x_i$	$\sum_{j=1}^J h(g_z(\mathbf{x}))$, where $h(s) = \log(1+s)$	$\hat{h}^L(g_z(\mathbf{x}))$	$\frac{1}{(L+1)2^{L+1}}$
IM	Instances $G = (V, E)$ of a directed graph, partitions $P_v^z \subset V$	$\sum_{i \in V} \frac{1}{N}\Big(1 - \prod_{u \in P_i^z}(1 - x_u)\Big)$	$h(g_z(\mathbf{x}))$ where $h(s) = \log(1+s)$	$\hat{h}^L(g_z(\mathbf{x}))$	$\frac{1}{(L+1)2^{L+1}}$
FL	Complete weighted bipartite graph $G = (V \cup V')$ weights $w_{i_\ell,z} \in [0,1]^{N \times \|z\|}$	$\sum_{\ell=1}^{N}(w_{i_\ell,z} - w_{i_{\ell+1},z})\Big(1 - \prod_{k=1}^{\ell}(1 - x_{i_k})\Big)$	$h(g_z(\mathbf{x}))$ where $h(s) = \log(1+s)$	$\hat{h}^L(g_z(\mathbf{x}))$	$\frac{1}{(L+1)2^{L+1}}$
CN	Graph $G = (V, E)$, service rates $\mu \in \mathbb{R}_+^{\|z\|}$, requests $r \in \mathcal{R}$, P_z path of r, arrival rates $\lambda \in \mathbb{R}_+^{\|\mathcal{R}\|}$	$\frac{1}{\mu_z}\sum_{r \in \mathcal{R}: z \in p^r} \lambda^r \prod_{k'=1}^{k_{p^r}(v)}(1 - x_{p_k^r, i^r})$	$h(g_z(\mathbf{0})) - h(g_z(\mathbf{x}))$ where $h(s) = s/(1-s)$	$\hat{h}^L(g_z(\mathbf{x}))$	$\frac{\bar{s}^{L+1}}{1-\bar{s}}$

where $\sum_i r_{i,z} = 1$. An example of such a value is the term frequency, i.e., the number of times the token appears in the document, divided by the document's length (in tokens). To be diverse, the summary should contain tokens that cover different subjects. To that end, if tokens are partitioned in to subjects, represented by a partition $\{P_j\}_{j=1}^J$ of V, the objective is given by $f(\mathbf{x}) = \mathbb{E}_z(f_z(\mathbf{x}))$ where $f_z(\mathbf{x}) = \sum_{j=1}^J h\left(\sum_{i \in V \cap P_j} r_{i,z} x_i\right)$, and $h(s) = \log(1+s)$ is a non-decreasing concave function. Intuitively, the concavity of h suppresses the selection of similar tokens (corresponding to the same subject), even if they have high value, thereby promoting diversity. Functions f_z (and, thereby, also f) are monotone and submodular, and we can construct polynomial approximators $\hat{f}_z^L$ for them as indicated in Table 1 by replacing h with its L^{th}-order Taylor approximation around $1/2$, given by:

$$\hat{h}^L(s) = \sum_{\ell=0}^{L} \frac{h^{(\ell)}(1/2)}{\ell!}(s - 1/2)^\ell. \tag{14}$$

This is because the composition of polynomial $\hat{f}_z^L$ with polynomial g_z in Table 1 is again a polynomial. We show in App. C that this estimator ensures that f indeed satisfies Asm. 3. Moreover, the estimator bias *decays exponentially* with degree L (see Tab. 1 and App. C), meaning that polynomial number of terms suffice to reduce the bias to a desired level. A partition matroid can be used with this objective to enforce that no more than k_ℓ sentences come from ℓ-th user, etc.

5.2 Influence Maximization (IM) [8, 14]

Given a directed graph $G = (V, E)$, we wish to maximize the expected fraction of nodes reached if we infect a set of nodes $S \subseteq V$ and the infection spreads via, e.g., the Independent Cascade (IC) model [14]. Adding a concave utility to the fraction can enhance the value of nodes reached in early stages. Formally, let z can be a random simulation trace of the IC model, and $P_v^z \subseteq V$ is the set of nodes reachable from v in a random simulation of the IC model. Then, the objective can be written as $f(\mathbf{x}) = \mathbb{E}_{z \sim P}[f_z(\mathbf{x})]$ where $f_z(\mathbf{x}) = h(g_z(\mathbf{x}))$, $h(s) = \log(1+s)$, and $g_z(\mathbf{x}) = \sum_{v \in V} \frac{1}{N}\big(1 - \prod_{i \in P_v^z}(1 - x_i)\big)$ is the number of infected nodes under seed set $\mathbf{x}$. Since functions $g_z : [0,1]^N \to [0,1]$ are multilinear, monotone submodular and $h : [0,1] \to \mathbb{R}$ is non-decreasing and concave, f satisfies Asm. 1 [23]. Again, we can construct $\hat{f}^L$ by replacing h by $\hat{h}^L$, given by Eq. (14). This again ensures that f indeed satisfies Asm. 3, and the estimator bias again decays exponentially (see Tab. 1 and App. C). Partition matroid constraints could be used in this setting to bound the number of seeds from some group (e.g., males/females, people in a zip code, etc.).

5.3 Facility Location (FL) [19]

Given a weighted bipartite graph $G = (V \cup V_z)$ and weights $w_{i,z} \in [0,1]$, $i \in V$, $z \in V_z$, we wish to maximize:

$$f(S) = \mathbb{E}_{z \sim P}\left[h(\max_{i \in S} w_{i,z})\right], \tag{15}$$

where $h(s) = \log(1+s)$. Intuitively, V and V' represent facilities and customers respectively and $w_{v,v'}$ is the utility of facility v for customer v'. The goal is to select a subset of facility locations $S \subset V$ to maximize the total utility, assuming every customer chooses the facility with the highest utility in the selection S; again, adding the concave function h adds diversity, favoring the satisfaction of customers that are not already covered. This too becomes a coverage problem by observing that [12]: $\max_{i \in S} w_{i,z} = \sum_{\ell=1}^{n}(w_{i_\ell,z} - w_{i_{\ell+1},z})\big(1 - \prod_{k=1}^{\ell}(1 - x_{i_k})\big)$, where, for a given $z \in V_z$, weights have been pre-sorted in a descending order as $w_{i_1,z} \geq \ldots \geq w_{i_n,z}$. and $w_{i_{n+1},j} \triangleq 0$. In a manner similar to Sec 5.2, we can show that this function again satisfies Asm. 1 and 3, using again the L^{th}-order Taylor approximation of h, given by Eq. (14); this will again lead to a bias that decays exponentially (see Tab. 1 and App. C). We can again optimize such an objective over arbitrary matroids, which can enforce, e.g., that no more than k facilities are selected from a geographic area or some other partition of V.

6 Experiments

We evaluate Alg. 1, with sampling and polynomial estimators over two well-known problem instances (influence maximization and facility location) with real and synthetic datasets. We summarize these setups in Tab. 2. For a more detailed overview of the datasets and experiment parameters, please refer to App. D . Our code is publicly accessible.[2]

Algorithms. We compare the performance of different estimators. These estimators are: (a) sampling estimator (SAMP) with $N = 1, 10, 20, 100$ and (b) polynomial estimator (POLY) with $L = 1, 2$.

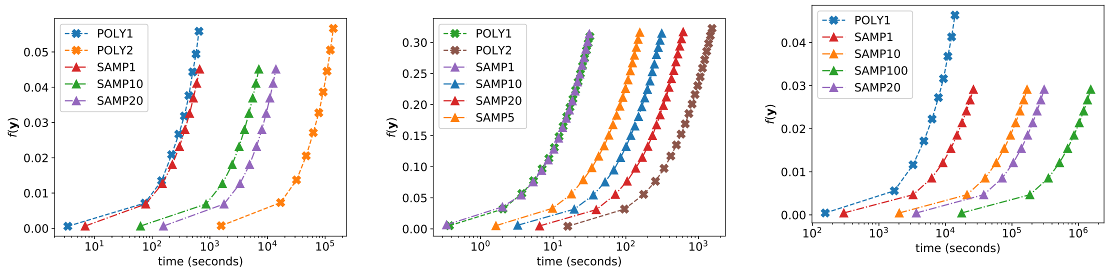
Metrics. We evaluate the performance of the estimators with their clock running time and via the maximum result ($\max f(\mathbf{y})$) obtained using the best available estimator for a given setting.

instance	dataset	$\|z\|$	$\|S\|$	$\|E\|$	m	k
IM	`SBPL`	20	400	914	4	1
IM	`ZKC`	20	34	78	2	3
FL	`MovieLens`	4000	6041	256	10	2

Table 2. Datasets and Experiment Parameters.

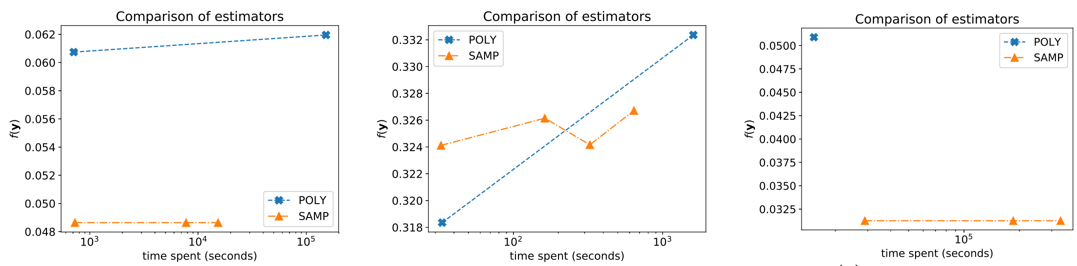
Results. The trajectory of the utility obtained at each iteration of the stochastic continuous greey algorithm $f(\mathbf{y})$ is plotted as a function of time in Fig. 1. In Fig. 1(a), we observe that polynomial estimators outperforms sampling estimators in terms of utility. Moreover, POLY1 runs 10 times faster than SAMP20 and runs in comparable time to SAMP1. In Fig. 1(b), POLY2 outperforms all estimators whereas POLY1 underperforms. Finally, in Fig. 1(c) we observe that POLY1 consistently outperforms sampling estimators.

[2] https://github.com/neu-spiral/StochSubMax



(a) SBPL (b) ZKC (c) MovieLens

Fig. 1. Trajectory of the FW algorithm. Utility of the function at the current $\mathbf{y}$ as a function of time is marked for every iteration.



(a) SBPL (b) ZKC (c) MovieLens

Fig. 2. Comparison of different estimators on different problems. Blue lines represent the performance of the POLY estimators and the marked points correspond to POLY1 and POLY2 respectively. Orange lines represent the performance of the SAMP estimators and the marked points correspond to SAMP1, SAMP10, SAMP20, SAMP100 respectively.

The final outcomes of the objective functions of the estimators are reported as a function of time in Fig. 2. In Fig. 2(a) and 2(b), POLY2 outperforms other estimators in terms of utility. Again in Fig. 2(a), POLY1 outperforms sampling estimators in terms of utility and runs in comparable time to SAMP1 while in Fig. 2(c), POLY1 outperforms sampling estimators both in terms of time and utility. Ideally, we would expect the performance of the estimators to improve as the degree of the polynomial or the number of samples increase. The examples where this is not always the case can be explained by the stochastic nature of the problem.

7 Conclusions

We show that polynomial estimators can improve existing stochastic submodular maximization methods by eliminating one of the two sources of randomness, particularly the one that stems from sampling. Investigating methodical ways to construct such polynomials can expand the applications of the proposed estimator appearing in this paper. Online versions of stochastic submodular optimization, where performance is characterized in terms of (approximate) regret, are also a possible future research direction.

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Appendix

A Proof of Lemma 2

We start by showing that the norm of the residual error vector of the estimator converges to 0. Recall that, by Asm. 3 the residual error of the polynomial estimation $\hat{f}_z^L(\mathbf{x})$ is bounded by $\varepsilon(L)$. Thus, for functions $f_z : \{0,1\}^n \to \mathbb{R}_+$ satisfying Asm. 3, we have that for all $\mathbf{y} \in [0,1]^n$,

$$|f_z(\mathbf{x}) - \hat{f}_z^L(\mathbf{x})| \leq \varepsilon_z(L). \tag{16}$$

Since $\lim_{L\to\infty} \varepsilon_z(L) = 0$ for all $\mathbf{y} \in [0,1]^n$, we get that

$$\lim_{L\to\infty} |f_z(\mathbf{y}) - \hat{f}_z^L(\mathbf{y})| \leq \lim_{L\to\infty} \varepsilon_z(L) = 0. \tag{17}$$

Moreover,

$$\begin{aligned}\left|\frac{\partial G_z(\mathbf{y})}{\partial y_i} - \frac{\widehat{\partial G_z^L}(\mathbf{y})}{\partial y_i}\right| &= \left|\mathbb{E}_{\mathbf{x}\sim\mathbf{y}}[f_z([\mathbf{x}]_{+i})] - \mathbb{E}_{\mathbf{x}\sim\mathbf{y}}[f_z([\mathbf{x}]_{-i})] - \mathbb{E}_{\mathbf{x}\sim\mathbf{y}}[\hat{f}_z^L([\mathbf{x}]_{+i})] + \mathbb{E}_{\mathbf{x}\sim\mathbf{y}}[\hat{f}_z^L([\mathbf{x}]_{-i})]\right| \\ &\leq \mathbb{E}_{\mathbf{x}\sim\mathbf{y}}[|f_z([\mathbf{x}]_{+i}) - \hat{f}_z^L([\mathbf{x}]_{+\mathbf{i}})|] + \mathbb{E}_{\mathbf{x}\sim\mathbf{y}}[|f_z([\mathbf{x}]_{-i}) - \hat{f}_z^L([\mathbf{x}]_{-i})|] \\ &\overset{(16)}{\leq} \mathbb{E}_{\mathbf{x}\sim\mathbf{y}}[\varepsilon_z(L))] + \mathbb{E}_{\mathbf{x}\sim\mathbf{y}}[\varepsilon_z(L))] = 2\varepsilon_z(L).\end{aligned}$$

Hence, we conclude that $\left\|\nabla G_z(\mathbf{y}) - \widehat{\nabla G_z^L}(\mathbf{y})\right\|_2 \leq 2\sqrt{n}\varepsilon_z(L)$. □

B Proof of Theorem 2

Proof. **Lemma 3.** *Consider Stochastic Continuous Greedy (SCG) outlined in Algorithm 1, and recall the definitions of the function G, the rank r, and $f_{\max} = \max_{i\in\{1,\dots,n\}} f(\{i\})$. If Assumption 2 is satisfied, then for $t = 0,\dots,T$ and for $j = 1,\dots,n$ we have*

$$\begin{aligned}\mathbb{E}\left[G(\mathbf{y}_{t+1})\right] \geq \mathbb{E}\left[G(\mathbf{y}_t)\right] + \frac{1}{T}\mathbb{E}\left[G(\mathbf{y}^*) - G(\mathbf{y}_t)\right] - \frac{f_{\max} r D^2}{2T^2} \\ - \frac{1}{2T}\left(4\beta_t D^2 - \frac{\mathbb{E}\left[\|\nabla G(\mathbf{y}_t) - \mathbf{d}_t\|^2\right]}{\beta_t}\right).\end{aligned} \tag{18}$$

Proof. According to the Taylor's expansion of the function G near the point $\mathbf{y}_t$ we can write

$$\begin{aligned}G(\mathbf{y}_{t+1}) &= G(\mathbf{y}_t) + \langle \nabla G(\mathbf{y}_t), \mathbf{y}_{t+1} - \mathbf{y}_t\rangle + \frac{1}{2}\langle \mathbf{y}_{t+1} - \mathbf{y}_t, \mathbf{H}(\tilde{\mathbf{y}}_t)(\mathbf{y}_{t+1} - \mathbf{y}_t)\rangle, \\ &= G(\mathbf{y}_t) + \frac{1}{T}\langle \nabla G(\mathbf{y}_t), \mathbf{v}_t\rangle + \frac{1}{2T^2}\langle \mathbf{v}_t, \mathbf{H}(\tilde{\mathbf{y}}_t)\mathbf{v}_t\rangle\end{aligned}$$

where $\tilde{\mathbf{y}}_t$ is a convex combination of $\mathbf{y}_t$ and $\mathbf{y}_t + \frac{1}{T}\mathbf{v}_t$ and $\mathbf{H}(\tilde{\mathbf{y}}_t) = \nabla^2 G(\tilde{\mathbf{y}}_t)$. Note here that the elements of the matrix $\mathbf{H}(\tilde{\mathbf{y}}_t)$ are less than the maximum marginal value (i.e. $\max_{i,j} |H_{i,j}(\tilde{\mathbf{y}}_t)| \leq$

$\max_{i\in\{1,\ldots,n\}} f(\{i\}) = f_{\max}$). Therefore, we can lower bound H_{ij} by $-f_{\max}$.

$$\begin{aligned}\langle \mathbf{v}_t, \mathbf{H}(\tilde{\mathbf{y}}_t)\mathbf{v}_t\rangle &= \sum_{j=1}^{n}\sum_{i=1}^{n} v_{i,t}v_{j,t}H_{ij}(\tilde{\mathbf{y}}_t)\\ &\geq -f_{\max}\sum_{j=1}^{n}\sum_{i=1}^{n} v_{i,t}v_{j,t}\\ &= -f_{\max}\left(\sum_{i=1}^{n} v_{i,t}\right)^2 = -f_{\max}r\|\mathbf{v}_t\|^2,\end{aligned}$$

where the last equality is because $\mathbf{v}_t$ is a vector with r ones and $n-r$ zeros. Replacing $\langle \mathbf{v}_t, \mathbf{H}(\tilde{\mathbf{y}}_t)\mathbf{v}_t\rangle$ by its lower bound $-f_{\max}r\|\mathbf{v}_t\|^2$ to obtain

$$G(\mathbf{y}_{t+1}) \geq G(\mathbf{y}_t) + \frac{1}{T}\langle \nabla G(\mathbf{y}_t), \mathbf{v}_t\rangle - \frac{f_{\max}r}{2T^2}\|\mathbf{v}_t\|^2. \tag{19}$$

Let $\mathbf{y}^*$ be the global maximizer within the constraint set $\mathcal{C}$. Since $\mathbf{v}_t$ is in the set $\mathcal{C}$, it follows from Assumption 2 that the norm $\|\mathbf{v}_t\|^2$ is bounded above by D^2. Apply this substitution and add and subtract the inner product $\frac{1}{T}\langle \mathbf{v}_t, \mathbf{d}_t\rangle$ to the right hand side of (19) to obtain

$$\begin{aligned}G(\mathbf{y}_{t+1}) &\geq G(\mathbf{y}_t) + \frac{1}{T}\langle \mathbf{v}_t, \mathbf{d}_t\rangle + \frac{1}{T}\langle \mathbf{v}_t, \nabla G(\mathbf{y}_t) - \mathbf{d}_t\rangle - \frac{f_{\max}rD^2}{2T^2}\\ &\geq G(\mathbf{y}_t) + \frac{1}{T}\langle \mathbf{y}^*, \mathbf{d}_t\rangle + \frac{1}{T}\langle \mathbf{v}_t, \nabla G(\mathbf{y}_t) - \mathbf{d}_t\rangle - \frac{f_{\max}rD^2}{2T^2}.\end{aligned} \tag{20}$$

Note that the second inequality in (20) holds since $\mathbf{v}_t \in \arg\max_{\mathbf{v}\in\mathcal{C}}\{\mathbf{d}_t^T\mathbf{v}\}$, we can write

$$\langle \mathbf{y}^*, \mathbf{d}_t\rangle \leq \max_{\mathbf{v}\in\mathcal{C}}\{\langle \mathbf{v}, \mathbf{d}_t\rangle\} = \langle \mathbf{v}_t, \mathbf{d}_t\rangle. \tag{21}$$

Now add and subtract the inner product $\langle \mathbf{y}^*, \nabla G(\mathbf{x}_t) - \mathbf{d}_t\rangle/T$ to the right hand side of (20) to get

$$G(\mathbf{y}_{t+1}) \geq G(\mathbf{y}_t) + \frac{1}{T}\langle \mathbf{y}^*, \nabla G(\mathbf{y}_t)\rangle + \frac{1}{T}\langle \mathbf{v}_t - \mathbf{y}^*, \nabla G(\mathbf{y}_t) - \mathbf{d}_t\rangle - \frac{f_{\max}rD^2}{2T^2}. \tag{22}$$

We further have $\langle \mathbf{y}^*, \nabla G(\mathbf{y}_t) - \mathbf{d}_t\rangle \geq G(\mathbf{y}^*) - G(\mathbf{y}_t)$; this follows from monotonicity of G as well as concavity of G along positive directions; see, e.g., [5]. Moreover, by Young's inequality we can show that the inner product $\langle \mathbf{v}_t - \mathbf{y}^*, \nabla G(\mathbf{y}_t) - \mathbf{d}_t\rangle$ is lower bounded by

$$\langle \mathbf{v}_t - \mathbf{y}^*, \nabla G(\mathbf{x}_t) - \mathbf{d}_t\rangle \geq -\frac{\beta_t}{2}\|\mathbf{v}_t - \mathbf{y}^*\|^2 - \frac{1}{2\beta_t}\|\nabla G(\mathbf{y}_t) - \mathbf{d}_t\|^2, \tag{23}$$

for any $\beta_t > 0$. By applying these substitutions into (22) we obtain

$$G(\mathbf{y}_{t+1}) \geq G(\mathbf{y}_t) + \frac{1}{T}(G(\mathbf{y}^*) - G(\mathbf{y}_t)) - \frac{f_{\max}rD^2}{2T^2} - \frac{1}{2T}\left(\beta_t\|\mathbf{v}_t - \mathbf{y}^*\|^2 - \frac{\|\nabla G(\mathbf{y}_t) - \mathbf{d}_t\|^2}{\beta_t}\right). \tag{24}$$

Replace $\|\mathbf{v}_t - \mathbf{y}^*\|^2$ by its upper bound $4D^2$ and compute the expected value of (24) to write

$$\begin{aligned}\mathbb{E}\left[G(\mathbf{y}_{t+1})\right] \geq \mathbb{E}\left[G(\mathbf{y}_t)\right] + \frac{1}{T}\mathbb{E}\left[G(\mathbf{y}^*) - G(\mathbf{y}_t)\right] - \frac{f_{\max} r D^2}{2T^2} \\ - \frac{1}{2T}\left(4\beta_t D^2 - \frac{\mathbb{E}\left[\|\nabla G(\mathbf{y}_t) - \mathbf{d}_t\|^2\right]}{\beta_t}\right).\end{aligned} \tag{25}$$

□

We need to introduce Lemma 7 to provide a bound for $\mathbb{E}\left[\|\nabla G(\mathbf{y}_t) - \mathbf{d}_t\|^2\right]$ and in order to prove Lemma 7, we need three new lemmas, Lemmas 4, 5, and 6, respectively.

Lemma 4. *Consider Stochastic Continuous Greedy (SCG) outlined in Algorithm 1 with iterates* $\mathbf{y}_t$*, and recall the definition of the multilinear extension function* G *in (2). If we define* r *as the rank of the matroid* $\mathcal{I}$ *and* $f_{\max} = \max_{i \in \{1,\ldots,n\}} f(i)$*, then the following holds*

$$|\nabla_j G(\mathbf{y}_{t+1}) - \nabla_j G(\mathbf{y}_t)| \leq f_{\max}\sqrt{r}\|\mathbf{y}_{t+1} - \mathbf{y}_t\|,$$

for $j = 1, \ldots, n$*.*

Proof. Same as the proof of Lemma 11 in [20]. □

Lemma 5. *The variance of the biased stochastic gradients* $\widehat{\nabla G^L_{z_t}(\mathbf{y}_t)}$ *is bounded above by* $\left(\sigma_0 + 2\sqrt{n}\varepsilon(L)\right)^2$*, i.e., for any vector* $\mathbf{y} \in \mathcal{C}$ *we can write*

$$\mathbb{E}\left[\left\|\nabla G(\mathbf{y}) - \widehat{\nabla G^L_z(\mathbf{y})}\right\|^2\right] \leq (1+\beta_0)\sigma_0^2 + \left(1 + \frac{1}{\beta_0}\right)2\sqrt{n}\varepsilon(L), \tag{26}$$

where the expectation is with respect to the randomness of $z \sim P$*.*

Proof. Adding and subtracting $\nabla G_{z_t}(\mathbf{y}_t)$ inside the norm, we obtain

$$\begin{aligned}\mathbb{E}\left[\left\|\nabla G(\mathbf{y}) - \widehat{\nabla G^L_{z_t}(\mathbf{y})}\right\|^2\right] &= \mathbb{E}\left[\left\|\nabla G(\mathbf{y}) - \nabla G_z(\mathbf{y}) + \nabla G_z(\mathbf{y}) - \widehat{\nabla G^L_z(\mathbf{y})}\right\|^2\right], \\ &= \mathbb{E}\bigg[\|\nabla G(\mathbf{y}) - \nabla G_z(\mathbf{y})\|^2 \\ &\quad + 2\left(\nabla G(\mathbf{y}) - \nabla G_z(\mathbf{y})\right)^T\left(\nabla G_z(\mathbf{y}) - \widehat{\nabla G^L_z(\mathbf{y})}\right) \\ &\quad + \left\|\nabla G_z(\mathbf{y}) - \widehat{\nabla G^L_z(\mathbf{y})}\right\|^2\bigg], \\ &= \mathbb{E}\left[\|\nabla G(\mathbf{y}) - \nabla G_z(\mathbf{y})\|^2\right] \\ &\quad + 2\mathbb{E}\left[\left(\nabla G(\mathbf{y}) - \nabla G_z(\mathbf{y})\right)^T\left(\nabla G_z(\mathbf{y}) - \widehat{\nabla G^L_z(\mathbf{y})}\right)\right] \\ &\quad + \mathbb{E}\left[\left\|\nabla G_z(\mathbf{y}) - \widehat{\nabla G^L_z(\mathbf{y})}\right\|^2\right].\end{aligned} \tag{27}$$

Using Young's inequality $\left(\langle \mathbf{a}, \mathbf{b}\rangle \leq \frac{1}{\beta}\frac{\|\mathbf{a}\|^2}{2} + \beta\frac{\|\mathbf{b}\|^2}{2}\right)$, also known as Peter-Paul inequality, to substitute the inner products with summations to obtain

$$\begin{aligned}
\mathbb{E}\left[\left\|\nabla G(\mathbf{y}) - \widehat{\nabla G^L_{z_t}(\mathbf{y})}\right\|^2\right] &= \mathbb{E}\left[\left\|\nabla G(\mathbf{y}) - \nabla G_z(\mathbf{y}) + \nabla G_z(\mathbf{y}) - \widehat{\nabla G^L_z(\mathbf{y})}\right\|^2\right], \\
&\leq \mathbb{E}\left[\|\nabla G(\mathbf{y}) - \nabla G_z(\mathbf{y})\|^2\right] \\
&\quad + \mathbb{E}\left[\beta_0 \|\nabla G(\mathbf{y}) - \nabla G_z(\mathbf{y})\|^2 + \frac{1}{\beta_0}\left\|\nabla G_z(\mathbf{y}) - \widehat{\nabla G^L_z(\mathbf{y})}\right\|^2\right] \\
&\quad + \mathbb{E}\left[\left\|\nabla G_z(\mathbf{y}) - \widehat{\nabla G^L_z(\mathbf{y})}\right\|^2\right], \\
&\leq (1+\beta_0)\mathbb{E}\left[\|\nabla G(\mathbf{y}) - \nabla G_z(\mathbf{y})\|^2\right] \\
&\quad + \left(1 + \frac{1}{\beta_0}\right)\mathbb{E}\left[\left\|\nabla G_z(\mathbf{y}) - \widehat{\nabla G^L_z(\mathbf{y})}\right\|^2\right].
\end{aligned} \tag{28}$$

Replacing $\mathbb{E}\left[\|\nabla G(\mathbf{y}) - \nabla G_z(\mathbf{y})\|^2\right]$ by its upper bound σ_0^2 and using the result of Lemma 2 to replace $\mathbb{E}\left[\left\|\nabla G_z(\mathbf{y}) - \widehat{\nabla G^L_z(\mathbf{y})}\right\|^2\right]$ by its upper bound $2\sqrt{n}\varepsilon(L)$, we obtain

$$\mathbb{E}\left[\left\|\nabla G(\mathbf{y}) - \widehat{\nabla G^L_z(\mathbf{y})}\right\|^2\right] \leq (1+\beta_0)\sigma_0^2 + \left(1 + \frac{1}{\beta_0}\right)2\sqrt{n}\varepsilon(L). \tag{29}$$

□

Lemma 6. *(Directly from [20]) Consider the scalars $b \geq 0$ and $c > 1$. Let ϕ_t be a sequence of real numbers satisfying*

$$\phi_t \leq \left(1 - \frac{c}{(t+t_0)^\alpha}\right)\phi_{t-1} + \frac{b}{(t+t_0)^{2\alpha}},$$

for some $0 \leq \alpha \leq 1$ and $t_0 \geq 0$. Then, the sequence ϕ_t converges to zero at the following rate

$$\phi_t \leq \frac{Q}{(t+t_0+1)^\alpha},$$

where $Q = \max\{\phi_0(t_0+1)^\alpha, b/(c-1)\}$.

Proof. Proof of the lemma can be found in the Appendix C of [20]. □

Lemma 7. *Consider Stochastic Continuous Greedy (SCG) outlined in Algorithm 1, and recall the definitions of the function G, the rank r, the upper bound σ_0 defined as in Thm. 2 and $f_{\max} = \max_{i\in\{1,\dots,n\}} f(\{i\})$. If Assumption 2 is satisfied and $\rho_t = \frac{4}{(t+8)^{2/3}}$, then for $t = 0, \dots, T$ and for $j = 1, \dots, n$ we have*

$$\mathbb{E}\left[\|\nabla G(\mathbf{y}_t) - \mathbf{d}_t\|^2\right] \leq \frac{Q}{(t+9)^{2/3}}, \tag{30}$$

where $Q = \max\{5\|\nabla G(\mathbf{y}_0 - \mathbf{d}_0)\|^2, 32\sigma_0^2 + 224\sqrt{n}\varepsilon(L) + 4f_{\max}^2 rD^2\}$.

Proof. Use the definition $\mathbf{d}_t = (1-\rho_t)\mathbf{d}_{t-1} + \rho_t \widehat{\nabla G^L_{z_t}(\mathbf{y}_t)}$ to write $\|\nabla G(\mathbf{y}_t) - \mathbf{d}_t\|^2$ as

$$\|\nabla G(\mathbf{y}_t) - \mathbf{d}_t\|^2 = \|\nabla G(\mathbf{y}_t) - (1-\rho_t)\mathbf{d}_{t-1} - \rho_t \widehat{\nabla G^L_{z_t}(\mathbf{y}_t)}\|^2. \tag{31}$$

Add and subtract the term $(1-\rho_t)\nabla G(\mathbf{y}_{t-1})$ to the right hand side of (31), regroup the terms and expand the squared term to obtain

$$\begin{aligned}\|\nabla G(\mathbf{y}_t) - \mathbf{d}_t\|^2 &= \|\nabla G(\mathbf{y}_t) - (1-\rho_t)\nabla G(\mathbf{y}_{t-1}) + (1-\rho_t)\nabla G(\mathbf{y}_{t-1}) \\ &\quad - (1-\rho_t)\mathbf{d}_{t-1} - \rho_t \widehat{\nabla G^L_{z_t}(\mathbf{y}_t)}\|^2 \\ &= \|\rho_t(\nabla G(\mathbf{y}_t) - \widehat{\nabla G^L_{z_t}(\mathbf{y}_t)}) + (1-\rho_t)(\nabla G(\mathbf{y}_t) - \nabla G(\mathbf{y}_{t-1})) \\ &\quad + (1-\rho_t)(\nabla G(\mathbf{y}_{t-1}) - \mathbf{d}_{t-1})\|^2 \\ &= \rho_t^2\|\nabla G(\mathbf{y}_t) - \widehat{\nabla G^L_{z_t}(\mathbf{y}_t)}\|^2 + (1-\rho_t)^2\|\nabla G(\mathbf{y}_t) - \nabla G(\mathbf{y}_{t-1})\|^2 \\ &\quad + (1-\rho_t)^2\|\nabla G(\mathbf{y}_{t-1}) - \mathbf{d}_{t-1}\|^2 \\ &\quad + 2\rho_t(1-\rho_t)(\nabla G(\mathbf{y}_t) - \widehat{\nabla G^L_{z_t}(\mathbf{y}_t)})^T(\nabla G(\mathbf{y}_t) - \nabla G(\mathbf{y}_{t-1})) \\ &\quad + 2\rho_t(1-\rho_t)(\nabla G(\mathbf{y}_t) - \widehat{\nabla G^L_{z_t}(\mathbf{y}_t)})^T(\nabla G(\mathbf{y}_{t-1}) - \mathbf{d}_{t-1}) \\ &\quad + 2(1-\rho_t)^2(\nabla G(\mathbf{y}_t) - \nabla G(\mathbf{y}_{t-1})^T(\nabla G(\mathbf{y}_{t-1}) - \mathbf{d}_{t-1}).\end{aligned} \tag{32}$$

Computing the conditional expectation $\mathbb{E}[(\cdot)|\mathcal{F}_t]$ for both sides we obtain

$$\begin{aligned}\mathbb{E}\left[\|\nabla G(\mathbf{y}_t) - \mathbf{d}_t\|^2|\mathcal{F}_t\right] &= \rho_t^2\mathbb{E}\left[\|\nabla G(\mathbf{y}_t) - \widehat{\nabla G^L_{z_t}(\mathbf{y}_t)}\|^2|\mathcal{F}_t\right] + (1-\rho_t)^2\|\nabla G(\mathbf{y}_t) - \nabla G(\mathbf{y}_{t-1})\|^2 \\ &\quad + (1-\rho_t)^2\|\nabla G(\mathbf{y}_{t-1}) - \mathbf{d}_{t-1}\|^2 \\ &\quad + 2\rho_t(1-\rho_t)\mathbb{E}\Big[(\nabla G(\mathbf{y}_t) - \widehat{\nabla G^L_{z_t}(\mathbf{y}_t)})^T(\nabla G(\mathbf{y}_t) - \nabla G(\mathbf{y}_{t-1}))\Big] \\ &\quad + 2\rho_t(1-\rho_t)\mathbb{E}\Big[(\nabla G(\mathbf{y}_t) - \widehat{\nabla G^L_{z_t}(\mathbf{y}_t)})^T(\nabla G(\mathbf{y}_{t-1}) - \mathbf{d}_{t-1})\Big] \\ &\quad + 2(1-\rho_t)^2(\nabla G(\mathbf{y}_t) - \nabla G(\mathbf{y}_{t-1}))^T(\nabla G(\mathbf{y}_{t-1}) - \mathbf{d}_{t-1}).\end{aligned} \tag{33}$$

Let's focus on the $\mathbb{E}\Big[(\nabla G(\mathbf{y}_t) - \widehat{\nabla G^L_{z_t}(\mathbf{y}_t)})^T(\nabla G(\mathbf{y}_t) - \nabla G(\mathbf{y}_{t-1}))\Big]$ term before moving further and call it A. By adding and subtracting $\nabla G_{z_t}(\mathbf{y}_t)$ inside $(\nabla G(\mathbf{y}_t) - \widehat{\nabla G^L_{z_t}(\mathbf{y}_t)})$ we obtain

$$\begin{aligned}A &= \mathbb{E}\Big[[(\nabla G(\mathbf{y}_t) - \nabla G_{z_t}(\mathbf{y}_t)) + (\nabla G_{z_t}(\mathbf{y}_t)) - \widehat{\nabla G^L_{z_t}(\mathbf{y}_t)}]^T(\nabla G(\mathbf{y}_t) - \nabla G(\mathbf{y}_{t-1}))\Big] \\ &= \mathbb{E}\Big[(\nabla G(\mathbf{y}_t) - \nabla G_{z_t}(\mathbf{y}_t))^T(\nabla G(\mathbf{y}_t) - \nabla G(\mathbf{y}_{t-1})) \\ &\quad + (\nabla G_{z_t}(\mathbf{y}_t) - \widehat{\nabla G^L_{z_t}(\mathbf{y}_t)})^T(\nabla G(\mathbf{y}_t) - \nabla G(\mathbf{y}_{t-1}))\Big] \\ &= \mathbb{E}\Big[(\nabla G(\mathbf{y}_t) - \nabla G_{z_t}(\mathbf{y}_t))^T(\nabla G(\mathbf{y}_t) - \nabla G(\mathbf{y}_{t-1}))\Big] \\ &\quad + \mathbb{E}\Big[(G_{z_t}(\mathbf{y}_t) - \widehat{\nabla G^L_{z_t}(\mathbf{y}_t)})^T(\nabla G(\mathbf{y}_t) - \nabla G(\mathbf{y}_{t-1}))\Big].\end{aligned} \tag{34}$$

Using the fact that $\nabla G_{z_t}(\mathbf{y}_t)$ is an unbiased estimator of the gradient $\nabla G(\mathbf{y}_t)$, i.e., $\mathbb{E}\left[\nabla G_{z_t}(\mathbf{y}_t)|\mathcal{F}_t\right] = \nabla G(\mathbf{y}_t)$, and replacing $(G_{z_t}(\mathbf{y}_t) - \widehat{\nabla G^L_{z_t}(\mathbf{y}_t)})$ with its upper bound $2\sqrt{n}\varepsilon(L)$ to obtain

$$A \leq \mathbb{E}\left[(2\sqrt{n}\varepsilon(L))^T(\nabla G(\mathbf{y}_t) - \nabla G(\mathbf{y}_{t-1}))\right]. \tag{35}$$

Applying a similar process to the $\mathbb{E}\left[(\nabla G(\mathbf{y}_t) - \nabla \widehat{G^L_{z_t}(\mathbf{y}_t)})^T(\nabla G(\mathbf{y}_{t-1}) - \mathbf{d}_{t-1})|\mathcal{F}_t\right]$ term and using Young's inequality $\left(\langle \mathbf{a}, \mathbf{b}\rangle \leq \frac{1}{\beta}\frac{\|\mathbf{a}\|^2}{2} + \beta\frac{\|\mathbf{b}\|^2}{2}\right)$, also known as Peter-Paul inequality, to substitute the inner products with summations to obtain

$$
\begin{aligned}
\mathbb{E}\left[\|\nabla G(\mathbf{y}_t) - \mathbf{d}_t\|^2|\mathcal{F}_t\right] \leq\ & \rho_t^2\mathbb{E}\left[\|\nabla G(\mathbf{y}_t) - \nabla \widehat{G^L_{z_t}(\mathbf{y}_t)}\|^2|\mathcal{F}_t\right] + (1-\rho_t)^2\|\nabla G(\mathbf{y}_t) - \nabla G(\mathbf{y}_{t-1})\|^2 \\
& + (1-\rho_t)^2\|\nabla G(\mathbf{y}_{t-1}) - \mathbf{d}_{t-1}\|^2 \\
& + (\rho_t - \rho_t^2)\mathbb{E}\left[\beta_1 2\sqrt{n}\varepsilon(L) + \frac{\|\nabla G(\mathbf{y}_t) - \nabla G(\mathbf{y}_{t-1})\|^2}{\beta_1}\right] \\
& + (\rho_t - \rho_t^2)\mathbb{E}\left[\beta_2 2\sqrt{n}\varepsilon(L) + \frac{\|\nabla G(\mathbf{y}_{t-1}) - \mathbf{d}_{t-1}\|^2}{\beta_2}\right] \\
& + (1-\rho_t)^2\left(\beta_3\|\nabla G(\mathbf{y}_t) - \nabla G(\mathbf{y}_{t-1})^T\|^2 + \frac{1}{\beta_3}\|\nabla G(\mathbf{y}_{t-1}) - \mathbf{d}_{t-1}\|^2\right), \\
=\ & \rho_t^2\mathbb{E}\left[\|\nabla G(\mathbf{y}_t) - \nabla \widehat{G^L_{z_t}(\mathbf{y}_t)}\|^2|\mathcal{F}_t\right] + (1-\rho_t)^2\|\nabla G(\mathbf{y}_t) - \nabla G(\mathbf{y}_{t-1})\|^2 \\
& + (1-\rho_t)^2\|\nabla G(\mathbf{y}_{t-1}) - \mathbf{d}_{t-1}\|^2 \\
& + (\rho_t - \rho_t^2)\left(\beta_1 2\sqrt{n}\varepsilon(L) + \frac{\|\nabla G(\mathbf{y}_t) - \nabla G(\mathbf{y}_{t-1})\|^2}{\beta_1}\right) \\
& + (\rho_t - \rho_t^2)\left(\beta_2 2\sqrt{n}\varepsilon(L) + \frac{\|\nabla G(\mathbf{y}_{t-1}) - \mathbf{d}_{t-1}\|^2}{\beta_2}\right) \\
& + (1-\rho_t)^2\left(\beta_3\|\nabla G(\mathbf{y}_t) - \nabla G(\mathbf{y}_{t-1})^T\|^2 + \frac{1}{\beta_3}\|\nabla G(\mathbf{y}_{t-1}) - \mathbf{d}_{t-1}\|^2\right), \\
=\ & \rho_t^2\mathbb{E}\left[\|\nabla G(\mathbf{y}_t) - \nabla \widehat{G^L_{z_t}(\mathbf{y}_t)}\|^2|\mathcal{F}_t\right] + \rho_t(1-\rho_t)(\beta_1+\beta_2)2\sqrt{n}\varepsilon(L) \\
& + \left((1-\rho_t)^2(1+\beta_3) + \frac{\rho_t(1-\rho_t)}{\beta_1}\right)\|\nabla G(\mathbf{y}_t) - \nabla G(\mathbf{y}_{t-1})\|^2 \\
& + \left((1-\rho_t)^2(1+\frac{1}{\beta_3}) + \frac{\rho_t(1-\rho_t)}{\beta_2}\right)\|\nabla G(\mathbf{y}_{t-1}) - \mathbf{d}_{t-1}\|^2.
\end{aligned} \tag{36}
$$

Since we assume that $\rho_t \leq 1$ we can replace all the $(1-\rho_t)^2$ terms by $(1-\rho_t)$. Applying this substitution and setting $\beta_1 = \rho_t$, $\beta_2 = 4$ and $\beta_3 = 4/\rho_t$, we get

$$
\begin{aligned}
\mathbb{E}\left[\|\nabla G(\mathbf{y}_t) - \mathbf{d}_t\|^2|\mathcal{F}_t\right] \leq\ & \rho_t^2\mathbb{E}\left[\|\nabla G(\mathbf{y}_t) - \nabla \widehat{G^L_{z_t}(\mathbf{y}_t)}\|^2|\mathcal{F}_t\right] + \rho_t(1-\rho_t)(\rho_t+4)2\sqrt{n}\varepsilon(L) \\
& + 2(1-\rho_t)\left(1+\frac{2}{\rho_t}\right)\|\nabla G(\mathbf{y}_t) - \nabla G(\mathbf{y}_{t-1})\|^2 \\
& + (1-\rho_t)\left(1+\frac{\rho_t}{2}\right)\|\nabla G(\mathbf{y}_{t-1}) - \mathbf{d}_{t-1}\|^2.
\end{aligned} \tag{37}
$$

Now using the inequalities $2(1-\rho_t)(1+(2/\rho_t)) \leq (4/\rho_t)$ and $(1-\rho_t)(1+(\rho_t/2)) \leq (1-(\rho_t/2))$ we obtain

$$\begin{aligned}\mathbb{E}\left[\|\nabla G(\mathbf{y}_t)-\mathbf{d}_t\|^2\right] &\leq \rho_t^2\mathbb{E}\left[\|\nabla G(\mathbf{y}_t)-\nabla \widehat{G^L_{z_t}(\mathbf{y}_t)}\|^2\right]+\rho_t(1-\rho_t)(\rho_t+4)2\sqrt{n}\varepsilon(L)\\ &+\frac{4}{\rho_t}\|\nabla G(\mathbf{y}_t)-\nabla G(\mathbf{y}_{t-1})\|^2+\left(1-\frac{\rho_t}{2}\right)\|\nabla G(\mathbf{y}_{t-1})-\mathbf{d}_{t-1}\|^2.\end{aligned} \tag{38}$$

Now we need two auxiliary lemmas (Lemma 4 & Lemma 5) to provide bounds for $\mathbb{E}\left[\|\nabla G(\mathbf{y}_t)-\nabla \widehat{G^L_{z_t}(\mathbf{y}_t)}\|^2\right]$ and $\|\nabla G(\mathbf{y}_t)-\nabla G(\mathbf{y}_{t-1})\|^2$, respectively.

The term $\mathbb{E}\left[\|\nabla G(\mathbf{y}_t)-\nabla \widehat{G^L_{z_t}(\mathbf{y}_t)}\|^2\right]$ can be bounded above by $(1+\beta_0)\sigma_0^2+\left(1+\frac{1}{\beta_0}\right)2\sqrt{n}\varepsilon(L)$ according to Lemma 5. Based on Assumption 2 and Lemma 4, we can also show that the squared norm $\|\nabla G(\mathbf{y}_t)-\nabla G(\mathbf{y}_{t-1})\|^2$ is upper bounded by $f_{\max}^2 rD^2/T^2$. Applying these substitutions yields

$$\begin{aligned}\mathbb{E}\left[\|\nabla G(\mathbf{y}_t)-\mathbf{d}_t\|^2\right] &\leq \rho_t^2(1+\beta_0)\sigma_0^2+\left(1+\frac{1}{\beta_0}+\rho_t(1-\rho_t)(\rho_t+4)\right)2\sqrt{n}\varepsilon(L)\\ &+\frac{4}{\rho_t}f_{\max}^2 rD^2/T^2+\left(1-\frac{\rho_t}{2}\right)\|\nabla G(\mathbf{y}_{t-1})-\mathbf{d}_{t-1}\|^2.\end{aligned}$$

We introduce one more Lemma 6 to bound $\mathbb{E}\left[\|\nabla G(\mathbf{y}_t)-\mathbf{d}_t\|^2\right]$ recursively.

Now define $\phi_t=\mathbb{E}\left[\|\nabla G(\mathbf{y}_t)-\mathbf{d}_t\|^2|\mathcal{F}_t\right]$ and set $\rho_t=\frac{4}{(t+8)^{2/3}}$ to obtain

$$\begin{aligned}\phi_t &\leq \left(1-\frac{2}{(t+8)^{2/3}}\right)\phi_{t-1}+\frac{16}{(t+8)^{4/3}}(1+\beta_0)\sigma_0^2\\ &+\left(1+\frac{1}{\beta_0}+\frac{4}{(t+8)^{2/3}}\left(1-\frac{4}{(t+8)^{2/3}}\right)\left(\frac{4}{(t+8)^{2/3}}+4\right)\right)2\sqrt{n}\varepsilon(L)\\ &+\frac{f_{\max}^2 rD^2(t+8)^{2/3}}{T^2}.\end{aligned}$$

Now use the conditions $8 \leq T$ and $t \leq T$ to replace $1/T$ by its upper bound $2/(t+8)$ and choose $\beta_0=1$:

$$\begin{aligned}\phi_t &\leq \left(1-\frac{2}{(t+8)^{2/3}}\right)\phi_{t-1}+\frac{32}{(t+8)^{4/3}}\sigma_0^2\\ &+\left(2+\frac{4}{(t+8)^{2/3}}\left(1-\frac{4}{(t+8)^{2/3}}\right)\left(\frac{4}{(t+8)^{2/3}}+4\right)\right)2\sqrt{n}\varepsilon(L)+\frac{4f_{\max}^2 rD^2}{(t+8)^{4/3}}.\end{aligned}$$

Now using the inequality $\left(2+\frac{4}{(t+8)^{2/3}}\left(1-\frac{4}{(t+8)^{2/3}}\right)\left(\frac{4}{(t+8)^{2/3}}+4\right)\right) \leq \frac{112}{(t+8)^{2/3}}$ for $t \geq 0$

$$\phi_t \leq \left(1-\frac{2}{(t+8)^{2/3}}\right)\phi_{t-1}+\frac{32\sigma_0^2+224\sqrt{n}\varepsilon(L)+4f_{\max}^2 rD^2}{(t+8)^{4/3}}.$$

Now using the result in Lemma 6, we obtain that

$$\phi_t \leq \frac{Q}{(t+9)^{2/3}}, \tag{39}$$

where $Q=\max\{5\|\nabla G(\mathbf{y}_0-\mathbf{d}_0)\|^2, 32\sigma_0^2+224\sqrt{n}\varepsilon(L)+4f_{\max}^2 rD^2\}$. □

Substitute $\mathbb{E}\left[\|\nabla G(\mathbf{y}_t) - \mathbf{d}_t\|^2\right]$ by its upper bound $Q/((t+9)^{2/3})$ according to the result of Lemma 7. Further, set $\beta_t = \frac{Q^{1/2}}{2D(t+9)^{1/3}}$ and regroup the resulted expression to obtain

$$\mathbb{E}\left[G(\mathbf{y}^*) - G(\mathbf{y}_{t+1})\right] \leq \left(1 - \frac{1}{T}\right)\mathbb{E}\left[G(\mathbf{y}^*) - G(\mathbf{y}_t)\right] + \frac{2DQ^{1/2}}{(t+9)^{1/3}T} + \frac{f_{\max} r D^2}{2T^2}. \tag{40}$$

By applying the inequality in (40) recursively for $t = 0, \ldots, T-1$ we obtain

$$\mathbb{E}\left[G(\mathbf{y}^*) - G(\mathbf{y}_T)\right] \leq \left(1 - \frac{1}{T}\right)^T \mathbb{E}\left[G(\mathbf{y}^*) - G(\mathbf{y}_0)\right] + \sum_{t=0}^{T-1} \frac{2DQ^{1/2}}{(t+9)^{1/3}T} + \sum_{t=0}^{T-1} \frac{f_{\max} r D^2}{2T^2}. \tag{41}$$

Note that we can write

$$\begin{aligned}\sum_{t=0}^{T-1} \frac{1}{(t+9)^{1/3}} &\leq \frac{1}{9^{1/3}} + \int_{t=0}^{T-1} \frac{1}{(t+9)^{1/3}} dt \\ &= \frac{1}{9^{1/3}} + \frac{3}{2}(t+9)^{2/3}\Big|_{t=T-1} - \frac{3}{2}(t+9)^{2/3}\Big|_{t=0} \\ &\leq \frac{3}{2}(T+8)^{2/3} \leq \frac{15}{2}T^{2/3}\end{aligned} \tag{42}$$

where the last inequality holds since $(T+8)^{2/3} \leq 5T^{2/3}$ for any $T \geq 1$. By simplifying the terms on the right hand side of (41) and using the inequality in (42) we can write

$$\mathbb{E}\left[G(\mathbf{y}^*) - G(\mathbf{y}_T)\right] \leq \frac{1}{e}\mathbb{E}\left[G(\mathbf{y}^*) - G(\mathbf{y}_0)\right] + \frac{15DQ^{1/2}}{T^{1/3}} + \frac{f_{\max} r D^2}{2T}. \tag{43}$$

Here, we use the fact that $G(\mathbf{y}_0) \geq 0$, and hence the expression in (43) can be simplified to

$$\mathbb{E}\left[G(\mathbf{y}_T)\right] \geq (1 - 1/e)\mathbb{E}\left[G(\mathbf{y}^*))\right] - \frac{15DQ^{1/2}}{T^{1/3}} - \frac{f_{\max} r D^2}{2T},$$

where $Q = \max\{5\|\nabla G(\mathbf{y}_0 - \mathbf{d}_0)\|^2, 32\sigma_0^2 + 224\sqrt{n}\varepsilon(L) + 4f_{\max}^2 r D^2\}$ and $K = Q^{1/2} = \max\{3\|\nabla G(\mathbf{y}_0 - \mathbf{d}_0)\|^2, \sqrt{16\sigma_0^2 + 224\sqrt{n}\varepsilon(L)} + 2\sqrt{r} f_{\max} D\}$. □

C Custom Biases for the Problems in Sec. 5

Estimator Bias for Summarization Problems

Theorem 3. *Assume a diversity reward function* $f:\ \{0,1\}^n \to \mathbb{R}_+$ *with* $h(s) = \log(1+s)$. *Then, consider the estimator* $\widehat{\nabla G_z^L}(\mathbf{y}_K)$ *given in (13) using* $\hat{f}_z^L(\mathbf{x})$, *the* L^{th} *Taylor polynomial of* $f(\mathbf{x})$ *around* $1/2$, *given by (14). Then, the bias of the estimator satisfies* $\left\|\nabla G_z(\mathbf{y}) - \widehat{\nabla G_z^L}(\mathbf{y})\right\|_2 \leq \frac{\sqrt{n}}{(L+1)2^L}$.

Proof. We begin by characterizing the residual error of the Taylor series of $h(s) = \log(1+s)$ around $1/2$:

Lemma 8. *Let* $\hat{h}^L(s)$ *be the* L^{th} *order Taylor approximation of* $h(s) = \log(1+s)$ *around* $1/2$, *given by (14). Then,* $\hat{f}_z^L(\mathbf{x}) = \hat{h}^L(g_z(\mathbf{x}))$, *satisfies Asm. 3, with:*

$$\varepsilon_z(L) = \frac{1}{(L+1)2^{L+1}}. \tag{44}$$

Proof. By the Lagrange remainder theorem,

$$\left|h(s)-\hat{h}^L(s)\right| = \left|\frac{h^{(L+1)}(s')}{(L+1)!}\left(s-\frac{1}{2}\right)^{L+1}\right|$$
$$= \left|\frac{(s-1/2)^{L+1}}{(L+1)\,(1+s')^{L+1}}\right|$$

for some s' between s and $1/2$. Since $s \in [0,1]$, (a) $|s-\frac{1}{2}| \leq \frac{1}{2}$, and (b) $s' \in [0,1]$. Hence $\left|h(s)-\hat{h}^L(s)\right| \leq \frac{1}{(L+1)2^{L+1}}$. □

To conclude the theorem, observe that:

$$\left\|\nabla G_z(\mathbf{y}) - \widehat{\nabla G_z^L}(\mathbf{y})\right\|_2 \leq 2\sqrt{n}\varepsilon(L) = \frac{\sqrt{n}}{(L+1)2^L}.$$

□

Estimator Bias for Influence Maximization Problems

Theorem 4. *For function $f: \{0,1\}^n \to \mathbb{R}_+$ that given by $f(\mathbf{x}) = \mathbb{E}_{z\sim P}[f_z(\mathbf{x})]$, where consider the estimator $\widehat{\nabla G_z^L}$ given in (13) using $\hat{f}_z^L$, the L^{th}-order Taylor approximation of f_z around $1/2$, given by (14). Then, the bias of estimator $\widehat{\nabla G_z^L}$ satisfies $\left\|\nabla G_z(\mathbf{y}) - \widehat{\nabla G_z^L}(\mathbf{y})\right\|_2 \leq \frac{\sqrt{n}}{(L+1)2^L}$.*

Proof. To prove the theorem, observe that for all $\mathbf{y} \in [0,1]^n$:

$$\left\|\nabla G_z(\mathbf{y}) - \widehat{\nabla G_z^L}(\mathbf{y})\right\|_2 \leq 2\sqrt{n}\varepsilon(L) = \frac{\sqrt{n}}{(L+1)2^L}.$$

□

Cache Networks (CN) [17] A Kelly cache network can be represented by a graph $G(V,E)$, $|E| = M$, service rates μ_j, $j \in E$, storage capacities c_v, $v \in V$, a set of requests $\mathcal{R}$, and arrival rates λ_r, for $r \in \mathcal{R}$. Each request is characterized by an item $i^r \in \mathcal{C}$ requested, and a path $p^r \subset V$ that the request follows. For a detailed description of these variables, please refer to [17]. Requests are forwarded on a path until they meet a cache storing the requested item. In steady-state, the traffic load on an edge (u,v) is given by

$$g_{(u,v)}(\mathbf{x}) = \frac{1}{\mu_{u,v}} \sum_{r\in\mathcal{R}:(v,u)\in p^r} \lambda^r \prod_{k'=1}^{k_{p^r}(v)} (1 - x_{p_k^r, i^r}). \tag{45}$$

where $\mathbf{x} \in \{0,1\}^{|V||\mathcal{C}|}$ is a vector of binary coordinates x_{vi} indicating if $i \in \mathcal{C}$ is stored in node $v \in V$. If s is the load on an edge, the expected total number of packets in the system is given by $h(s) = \frac{s}{1-s}$. Then using the notation $z = (u,v) \in E$ to index edges, the expected total number of packets in the system in steady state can indeed be written as $\mathbb{E}_{z\sim P}\left[h(g_z(\mathbf{x}))\right]$ [17]. Mahdian et al. maximize the *caching gain* $f:\{0,1\}^{|V||\mathcal{C}|}\to\mathbb{R}_+$ as

$$f(\mathbf{x})=\mathbb{E}_{z\sim P}\left[h(g_z(\mathbf{0}))-h(g_z(\mathbf{x}))\right] \tag{46}$$

subject to the capacity constraints in each class. The caching gain $f(\mathbf{x})$ is monotone and submodular, and the capacity constraints form a partition matroid [17]. Moreover, $h(s)=\frac{s}{1-s}$ can be approximated within arbitrary accuracy by its L^{th}-order Taylor approximation around 0, given by:

$$\hat{h}^L(s)=\sum_{\ell=1}^{L}s^\ell \tag{47}$$

We show in App. C that this estimator ensures that f indeed satisfies Asm. 3. Proof of this lemma can be found in App. C. Furthermore, we bound the estimator bias appearing in Thm. 2 as follows:

Theorem 5. *Assume a caching gain function $f:\ \{0,1\}^{|V||\mathcal{C}|}\to\mathbb{R}_+$ that is given by (46). Then, consider Algorithm 1 in which $\nabla G(\mathbf{y}_K)$ is estimated via the polynomial estimator given in (13) where $\hat{f}_z^L(\mathbf{x})$ is the L^{th} Taylor polynomial of $f(\mathbf{x})$ around 0. Then, the bias of the estimator is bounded by*

$$\left\|\nabla G_z(\mathbf{y})-\widehat{\nabla G_z^L}(\mathbf{y})\right\|_2\leq 2\sqrt{|V||\mathcal{C}|}\frac{\bar{s}^{L+1}}{1-\bar{s}}, \tag{48}$$

where $\bar{s}<1$ is the largest load among all edges when caches are empty.

Proof. **Lemma 9.** *Let $\hat{h}^L(s)$ be the L^{th} Taylor polynomial of $h(s)=\frac{s}{1-s}$ around 0. Then, $h(s)$ and its polynomial estimator of degree L, $\hat{h}^L(s)$, satisfy Asm. 3 where*

$$\varepsilon(L)=\frac{\bar{s}^{L+1}}{1-\bar{s}}. \tag{49}$$

Proof. L^{th} Taylor polynomial of $h(s)$ around 0 is

$$\hat{h}_L(s)=\sum_{l=0}^{L}\frac{h^{(\ell)}(0)}{\ell!}s^\ell=\sum_{\ell=1}^{L}s^\ell \tag{50}$$

where $h^{(\ell)}(s)=\frac{\ell!}{(1-s)^{\ell+1}}$ for $h(s)=\frac{s}{1-s}$.

$$\begin{aligned}h(s)=\frac{s}{1-s}&=\sum_{\ell=1}^{\infty}s^\ell=\sum_{\ell=1}^{L}s^\ell+\sum_{\ell=L+1}^{\infty}s^\ell\\&=\sum_{\ell=1}^{L}s^\ell+s^L\sum_{\ell=1}^{\infty}s^\ell=\sum_{\ell=1}^{L}s^\ell+\frac{s^{L+1}}{1-s}\end{aligned}$$

Then, the bias of the Taylor Series Estimation around 0 becomes:

$$\left|\frac{s}{1-s}-\sum_{\ell=1}^{L}s^\ell\right|=\frac{s^{L+1}}{1-s}\leq\frac{\bar{s}^{L+1}}{1-\bar{s}}=\varepsilon(L).$$

for all $s\in[0,\bar{s}]$ where $\bar{s}=\max_{z\sim P}s_z$. □

Since $\lim_{L\to\infty}\frac{\bar{s}^{L+1}}{1-\bar{s}}=0$, for all $\bar{s}\in[0,1)$, Taylor approximation gives an approximation guarantee for maximizing the queue size function by Asm. 3, where the error of the approximation is given by Lem. 2 as

$$\left\|\nabla G_z(\mathbf{y})-\widehat{\nabla G_z^L}(\mathbf{y})\right\|_2\leq 2\sqrt{|V||\mathcal{C}|}\varepsilon(L)=2\sqrt{|V||\mathcal{C}|}\frac{s^{L+1}}{1-s}.$$

Then, $\varepsilon(L)\leq 2\sqrt{|V||\mathcal{C}|}\frac{\bar{s}^{L+1}}{1-\bar{s}}$. □

□

D Experimentation Details

Synthetic Datasets. We generate directed bipartite graph instances with number of instances $|z| = 1$, 5, 10, 100 and number of nodes $n = 200$, 400, 1000. Nodes are equally divided into left (V_1) and bottom (V_2) nodes where $|V_1| = |V_2|$. The seeds are always selected from V_1 and edges are placed between V_1 and V_2 so that the degrees of the nodes either follow a uniform or power law distribution.
Real Datasets. We use two real-world datasets `Zachary Karate Club` (`ZKC`) [27] and `MovieLens` [10].
Influence Maximization. We experiment on three synthetic datasets and one real dataset. For synthetic data, We generate directed bipartite graph instances with number of instances $|z| = 1$, 5, 10, 100 and number of nodes $n = 200$, 400, 1000. Nodes are equally divided into left (V_1) and bottom (V_2) nodes where $|V_1| = |V_2|$. The seeds are always selected from V_1 and edges are placed between V_1 and V_2 so that the degrees of the nodes follow power law (`SyntheticBipartitePowerlaw`) distribution. We construct a partition matroid of $m = 4$ equal-size partitions of V_1 and set $k = 1$ or 2 elements from each partition. The real dataset is the `Zachary Karate Club` (`ZKC`) [27]. We generate $|z| = 20$ cascades following the independent cascade model [14] using `Network Diffusion Library` [24]. The probability for each node to influence its neighbors is set to $p = 0.5$. We divide the dataset into two partitions following its existing labels and set $k = 3$.
Facility Location. For *facility location* problems, we experiment on the `MovieLens` dataset [10]. it has 1M ratings from $n = 6041$ users for $|z| = 4000$ movies. We treat movies as facilities, users as customers and ratings as $w_{i,j}$. We divide the movies into $m = 10$ partitions based on the first genre name listed for each movie and finally we set $k = 2$.